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IMPROVED FINITE DÍFFERENCE FORMULAS

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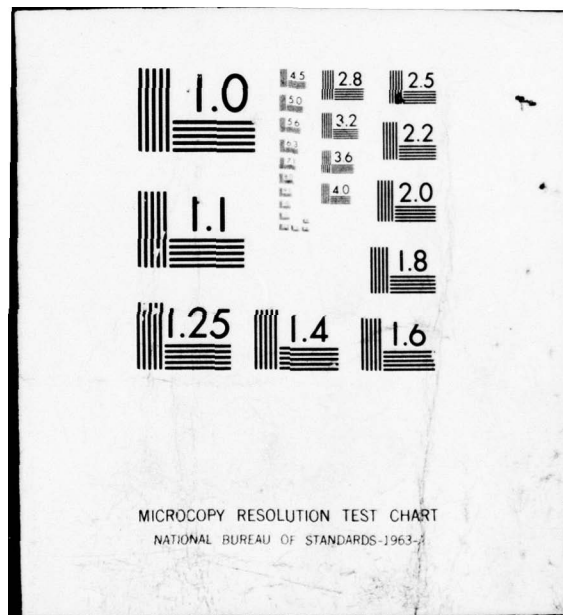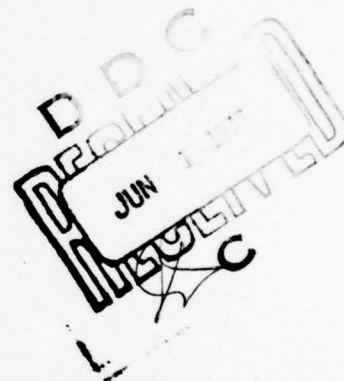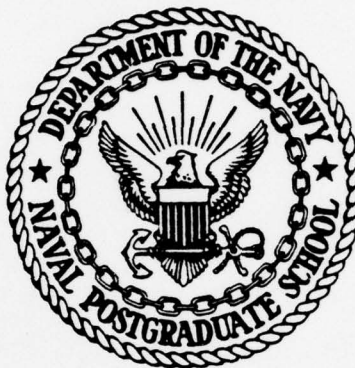BOUNDARY VALUE PROBLEMS

T. H. GAWAIN

AND

R. E. BALL

MAY 1977

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NAVAL POSTGRADUATE SCHOOL

Monterey, California

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The improvement in accuracy that can be achieved with the revised formulas is illustrated by two representative examples. The revised formulas are shown to reduce the overall error of the numerical solution by a factor of about five in a typical case.

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SUMMARY

This paper deals with the numerical solution of linear differential equations of fourth order by finite differences. It points out significant (but usually overlooked) errors which result from the conventional method of imposing the boundary conditions in such problems. Revised finite difference formulas are derived which apply near the boundaries and which eliminate the above errors.

Three commonly encountered boundary conditions are considered. These correspond, in the terminology of beam analysis, to a clamped end, to a simply supported end and to a free end.

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Table of Contents

1. Introduction

Let $\phi(x)$ denote an unknown function which is governed by a linear differential equation of the following form

$$D^4\phi + f_3(x) D^3\phi + f_2(x) D^2\phi + f_1(x) D\phi + f_o(x) \phi = g(x) \qquad (1.1)$$

where the coefficients $f_3(x)$, $f_2(x)$, $f_1(x)$, $f_o(x)$ and the forcing function $g(x)$ are all known functions.

We seek an approximate numerical solution of Eq. (1.1) by finite differences which satisfies appropriate boundary conditions at $x = 0$ and at $x = \ell$. Three commonly encountered boundary conditions are considered in this paper, and are labelled below according to the terminology used in beam analysis, namely,

1) Clamped End

$$\left.\begin{array}{r} \phi = 0 \\ D\phi = 0 \end{array}\right\} \qquad \text{at } x = 0 \text{ or at } x = \ell \qquad (1.2)$$

2) Simply Supported End

$$\left.\begin{array}{r} \phi = 0 \\ D^2\phi = 0 \end{array}\right\} \qquad \text{at } x = 0 \text{ or at } x = \ell \qquad (1.3)$$

3) Free End

$$\left.\begin{array}{r} D^2\phi = 0 \\ D^3\phi = 0 \end{array}\right\} \qquad \text{at } x = 0 \text{ or at } x = \ell \qquad (1.4)$$

The boundary conditions imposed at the two ends may be like or unlike. There is no restriction, except of course that if one end be free, the other is normally clamped to ensure that the beam configuration remains stable under arbitrary loading.

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In the finite difference analysis, the domain $0 \leq x \leq \ell$ is subdivided into N' equal intervals each of width

$$h = \frac{\ell}{N'} \tag{1.5}$$

If we include the two end points, the above domain contains $(N' + 1)$ stations. The function ϕ is now represented by its discrete values ϕ_i ($i = 1, 2, 3, \ldots$) at these stations. It is clear from Eqs. (1.2) and (1.3), however, that the function vanishes at either or at both of the end points depending on the particular boundary conditions that happen to apply. In any case it is convenient in the present context to let N denote the actual number of stations at which ϕ_i is initially unknown. For a beam whose ends are either clamped or simply supported, $N = N' - 1$. For a beam free at one end and clamped at the other, $N = N'$.

It is also convenient to designate the stations at which ϕ is initially unknown, from left to right, by index $i = 1, 2, 3, \ldots N$. For a beam free at the left end, index $i = 1$ denotes the actual free end itself, at location $x = 0$. For a beam *clamped or simply supported* at the left end, however, index $i = 1$ denotes the first station inboard of the left end, at location $x = h$. This convention simplifies the indexing of the final matrix equations so that they always run from $i = 1$ to $i = N$ inclusive.

To determine the unknowns $\phi_1, \phi_2, \phi_3, \ldots \phi_N$, we rewrite Eq. (1.1) in finite difference form for the ith station, then require that the resulting equation be satisfied for each of the

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stations $i = 1, 2, 3, \ldots N$. This produces N simultaneous
equations in N unknowns which suffices to establish the required
solution. The actual manipulations are handled most expediently
in matrix format.

Except for the stations at or immediately adjacent to the
boundaries, the various derivatives in Eq. (1.1) are usually
approximated by the four conventional central difference formulas
summarized in Eqs. (1) through (4) of Table I.

These formulas are based on approximating the function ϕ
in the vicinity of arbitrary station i by a truncated series of
the form

$$(\phi - \phi_i) = D\phi_i \, (x-x_i) + D^2\phi_i \, \frac{(x-x_i)^2}{2!} + D^3\phi_i \, \frac{(x-x_i)^3}{3!} + \ldots \quad (1.6)$$

The number of terms retained in this series depends on the accuracy
required in the final difference formulas. Eq. (1.6) is applied
to a number of contiguous stations symmetrically disposed on both
sides of station i. This yields a set of simultaneous equations
which can be solved for the initially unknown coefficients $D\phi_i$,
$D^2\phi_i$, $D^3\phi_i$, $\ldots$ of the series thus yielding the required finite
difference formulas. These formulas are represented by the
bracketed expressions in Eqs. (1) through (4) of Table I.

By re-introducing an additional term into the original series
and retracing the above solution, an estimate of the truncation
error is obtained. This is represented by the final term, that
is, the term outside the brackets, in Eqs. (1) through (4) of Table
I. Theoretically, the total truncation error could be represented

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Table I   Conventional Finite Difference Formulas

Central Differences

$$D\phi_i \doteq \frac{1}{h}\left[0 - \frac{1}{2}\phi_{i-1} + 0 + \frac{1}{2}\phi_{i+1} + 0\right] - \frac{1}{6}h^2 D^3\phi_i + \ldots \tag{1}$$

$$D^2\phi_i \doteq \frac{1}{h^2}\left[0 + \phi_{i-1} - 2\phi_i + \phi_{i+1} + 0\right] - \frac{1}{12}h^2 D^4\phi_i + \ldots \tag{2}$$

$$D^3\phi_i \doteq \frac{1}{h^3}\left[-\frac{1}{2}\phi_{i-2} + \phi_{i-1} + 0 - \phi_{i+1} + \frac{1}{2}\phi_{i+2}\right] - \frac{1}{4}h^2 D^5\phi_i + \ldots \tag{3}$$

$$D^4\phi_i \doteq \frac{1}{h^4}\left[\phi_{i-2} - 4\phi_{i-1} + 6\phi_i - 4\phi_{i+1} + \phi_{i+2}\right] - \frac{1}{6}h^2 D^6\phi_i + \ldots \tag{4}$$

Clamped End

$$D\phi_1 \doteq \frac{1}{h}\left[0 + \frac{1}{2}\phi_2 + 0\right] \qquad\qquad -\frac{1}{6}h^2 D^3\phi_1 + \ldots \tag{5}$$

$$D^2\phi_1 \doteq \frac{1}{h^2}\left[-2\phi_1 + \phi_2 + 0\right] \qquad\qquad -\frac{1}{12}h^2 D^4\phi_1 + \ldots \tag{6}$$

$$D^3\phi_1 \doteq \frac{1}{h^3}\left[\frac{1}{2}\phi_1 - \phi_2 + \frac{1}{2}\phi_3\right] \qquad\qquad -\frac{1}{6}D^3\phi_0 + \ldots \tag{7}$$

$$D^4\phi_1 \doteq \frac{1}{h^4}\left[7\phi_1 - 4\phi_2 + \phi_3\right] \qquad\qquad -\frac{1}{3}\frac{D^3\phi_0}{h} + \ldots \tag{8}$$

Table I Cont'd

Simply Supported End

$$D\phi_1 \doteq \frac{1}{h}\left[0 + \frac{1}{2}\phi_2 + 0\right] \qquad\qquad -\frac{1}{6}h^2 D^3\phi_1 + \ldots \qquad (9)$$

$$D^2\phi_1 \doteq \frac{1}{h^2}\left[-2\phi_1 + \phi_2 + 0\right] \qquad\quad -\frac{1}{12}h^2 D^4\phi_1 + \ldots \qquad (10)$$

$$D^3\phi_1 \doteq \frac{1}{h^3}\left[\frac{1}{2}\phi_1 - \phi_2 + \frac{1}{2}\phi_3\right] \qquad -\frac{1}{24}h D^4\phi_0 + \ldots \qquad (11)$$

$$D^4\phi_1 \doteq \frac{1}{h^4}\left[+5\phi_1 - 4\phi_2 + \phi_3\right] \qquad +\frac{1}{12}D^4\phi_0 + \ldots \qquad (12)$$

Free End

At station i = 1

$$D\phi_1 \doteq \frac{1}{h}\left[-\phi_1 + \phi_2 + 0\right] \qquad\qquad -\frac{1}{6}h^2 D^3\phi_1 + \ldots \qquad (13)$$

$$D^2\phi_1 \doteq 0 \qquad (14)$$

$$D^3\phi_1 = 0 \qquad (15)$$

$$D^4\phi_1 \doteq \frac{1}{h^4}\left[2\phi_1 - 4\phi_2 + 2\phi_3\right] \qquad -\frac{1}{6}D^4\phi_1 + \ldots \qquad (16)$$

At station i = 2

$$D\phi_2 \doteq \frac{1}{h}\left[\frac{1}{2}\phi_1 + 0 + \frac{1}{2}\phi_3 + 0\right] \qquad -\frac{1}{6}h^2 D^3\phi_2 + \ldots \qquad (17)$$

$$D^2\phi_2 \doteq \frac{1}{h^2}\left[\phi_1 - 2\phi_2 + \phi_3 + 0\right] \qquad -\frac{1}{12}h^2 D^4\phi_2 + \ldots \qquad (18)$$

$$D^3\phi_2 \doteq \frac{1}{h^3}\left[0 + \frac{1}{2}\phi_2 - \phi_3 + \frac{1}{2}\phi_4\right] \qquad -\frac{1}{24}h D^4\phi_1 + \ldots \qquad (19)$$

$$D^4\phi_2 \doteq \frac{1}{h^4}\left[-2\phi_1 + 5\phi_2 - 4\phi_3 + \phi_4\right] + \frac{1}{12}D^4\phi_1 + \ldots \qquad (20)$$

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by an infinite series of which the quantity shown is merely the leading term.

Notice that all four of the conventional central difference formulas shown in Table I have truncation errors of order h^2. Additional terms of the series which represent the truncation error are not given in the table but they can be shown to involve steadily ascending powers of h, that is, $h^3, h^4, h^5, \ldots$ etc. Hence in the limit of very small mesh size h, the higher order terms become relatively negligible and the single term shown becomes itself an adequate approximation to the overall truncation error.

Also notice that for a truncation error of order h^2, the central difference formulas for $D\phi_i$ and $D^2\phi_i$ require a band width of only three, that is, they involve the values of ϕ at only the three successive stations $(i-1)$, i and $(i+1)$. To achieve the same order of truncation error for $D^3\phi_i$ and $D^4\phi_i$ on the other hand requires a band width of five.

When reduced to matrix format, the basic relation Eq. (1,1) has a band width equal to the widest band of any derivative which appears in it. For a given order of truncation error, the derivative of highest order requires the greatest band width. Thus for a truncation error of order h^2, the final calculation matrix representing Eq. (1.1) has a band width of five; it is a penta-diagonal matrix.

The first four formulas in Table I may be applied routinely at interior stations, but special problems arise near the boundaries. It suffices here to illustrate the problem for the left end. Consider the case where the left end is either clamped or

simply supported. Then at station $i=1$ immediately adjacent
to the end in question, the conventional central difference
expressions for $D\phi$, and $D^2\phi$, involve the three quantities
ϕ_0, ϕ_1, and ϕ_2. The corresponding central difference ex-
pressions for $D^3\phi$, and $D^4\phi$ involve the five quantities
ϕ_{-1}, ϕ_0, ϕ_1, ϕ_2 and ϕ_3. The quantities ϕ_1, ϕ_2 and ϕ_3
entail no problems. The quantity ϕ_0 vanishes for either a
clamped end or a simply supported end and therefore involves no
difficulties. The problem that now arises, however, is that the
quantity ϕ_{-1} lies outside the normal domain of integration
$o \leq x \leq \ell$, and is in fact undefined. Thus $D^3\phi$, and $D^4\phi$,
cannot be evaluated from the usual central difference formulas
until this difficulty be resolved.

A similar problem arises also if the left end be free. Re-
call from the labelling convention adopted earlier that for a
free end, the actual end of the beam at location $x = 0$ is
denoted by station $i = 1$. Application of the conventional
central difference formulas at this station involves the five
quantities ϕ_{-1}, ϕ_0, ϕ_1, ϕ_2, ϕ_3. Application of the conven-
tional central difference formulas at the next station $i=2$
involves the five quantities ϕ_0, ϕ_1, ϕ_2, ϕ_3, ϕ_4. Once again,
however, the quantities ϕ_{-1} and ϕ_0 are initially undefined
and the conventional equations cannot be applied until this
double indeterminacy be resolved.

It is apparent from considerations of symmetry that
corresponding questions arise also at the right end of the span.

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Fortunately, there exists a generally accepted convention for resolving these questions. The rationale of this conventional solution, along with a careful analysis of its limitations, is presented in the next section. The results are in some respects surprising.

2. The Conventional Boundary Formulas and Their Limitations

The conventional solution for the left end clamped can be found by making the following substitutions in Eq. (1) of Table I, namely,

$$i = 0$$
$$\phi_0 = 0$$ (2.1)
$$D\phi_0 = 0$$

The resulting relation can then be solved for ϕ_{-1} as follows

$$\phi_{-1} = \phi_{+1} - \frac{1}{3} h^3 D^3 \phi_0 + \ldots$$ (2.2)

We can now evaluate Eqs. (1) through (4) of Table I at station $i = 1$ making use of the foregoing substitutions for ϕ_0 and ϕ_{-1}. In this way we obtain the conventional finite difference formulas for a *clamped end as summarized in Eqs. (5) through (8) of the* table.

A similar process may be used for the simply supported end. Thus we make the following substitutions in Eq. (2) of Table I, namely,

$$i = 0$$
$$\phi_0 = 0$$ (2.3)
$$D^2\phi_0 = 0$$

The resulting relation can then be solved for ϕ_{-1} as follows

$$\phi_{-1} = -\phi_{+1} + \frac{1}{12} h^4 D^4 \phi_0 + \ldots$$ (2.4)

2-1

We can again evaluate Eqs. (1) through (4) of Table I at station $i = 1$ by making use of the above substitutions for ϕ_0 and ϕ_{-1} . In this way we obtain the conventional finite difference formulas for a simply supported end as summarized by Eqs. (9) through (12) of the table.

The conventional solution for the left end free can be found by making the following substitutions in Eqs. (2) and (3) of Table I, namely,

$$i = 1$$
$$D^2\phi_1 = 0$$
$$D^3\phi_1 = 0$$

(2.5)

The resulting pair of equations can then be solved simultaneously for ϕ_0 and ϕ_1 . This gives

$$\phi_0 = (2\phi_1 - \phi_2 + 0) + \frac{1}{12} h^4 D^4\phi_1 + \ldots$$

(2.6)

$$\phi_{-1} = (4\phi_1 - 4\phi_2 + \phi_3) + \frac{1}{6} h^4 D^4\phi_1 + \ldots$$

(2.7)

We can now evaluate Eqs. (1) through (4) of Table I at station $i = 1$ in the usual way, by making use of Eqs. (2.6) and (2.7) to eliminate the quantities ϕ_0 and ϕ_1 wherever they occur. In this way we obtain the conventional finite difference formulas at station $i = 1$ for a free end as summarized in Eqs. (13) through (16) of the table.

Notice that for a free end, special formulas are required not only at station $i = 1$ but also at station $i = 2$. At the latter point the quantity ϕ_{-1} is no longer involved so that Eq. (2.7) is no longer needed. But Eq. (2.6) is still needed to eliminate ϕ_0

wherever it occurs. In this way we obtain the conventional finite
difference formulas at station i = 2 for a free end as summarized
in Eqs. (17) through (20) of Table I.

It is usually assumed that the conventional boundary formulas
all have truncation errors of order h^2 because they are derived
from central difference formulas having truncation errors of order
h^2 . Unfortunately, the results listed in Table I show that this
is not necessarily the case. Notice that if the truncation error
terms be omitted from Eqs. (2.2), (2.4), (2.6) and (2.7) by over-
sight, the conventional difference formulas will all seem to be of
order h^2 . This point is easily overlooked and it probably
accounts for the widespread misunderstanding concerning the true
truncation errors of the conventional finite difference formulas.
An important aim of this paper is to call attention to this common
oversight.

It is clear from Table I that some of the conventional formulas
have truncation errors of order h , h^o or even h^{-1} ! Such errors
are excessive and are inconsistent with the order of error involved
in the rest of the calculation matrix. They can be expected to
increase unnecessarily the overall error of the numerical solution.
Sample calculations presented later in this paper confirm that
this is indeed the case. It is shown that revision of these faulty
expressions greatly reduces the overall error of the final solu-
tion.

The method of making the needed revisions, and the results of
these revisions, are summarized in the next section.

3. <u>Revised Finite Difference Equations of Consistent Truncation</u>
   <u>Error</u>

Fortunately, the method of developing revised finite difference
formulas where needed is straight forward, if sometimes rather
tedious. The function ϕ in the vicinity of the end is repre-
sented by a truncated power series. The form of the series must
be such as to satisfy the boundary conditions of interest as
previously summarized in Eqs. (1.2), (1.3) or (1.4)

Thus for a clamped end at $x = 0$ we set

$$\phi = D^2\phi_0 \frac{x^2}{2!} + D^3\phi_0 \frac{x^3}{3!} + D^4\phi_0 \frac{x^4}{4!} + D^5\phi_0 \frac{x^5}{5!} + \ldots \tag{3.1}$$

For a simply supported end at $x = 0$ we set

$$\phi = D\phi_0 x + D^3\phi_0 \frac{x^3}{3!} + D^4\phi_0 \frac{x^4}{4!} + D^5\phi_0 \frac{x^5}{5!} + \ldots \tag{3.2}$$

For a free end at $x = 0$ we set

$$(\phi - \phi_1) = D\phi_1 x + D^4\phi_1 \frac{x^4}{4!} + D^5\phi_1 \frac{x^5}{5!} + \ldots \tag{3.3}$$

The number of terms retained in the series depends as usual
on the order of the derivative to be estimated and the truncation
error of the estimate. The truncated series expression is applied
to a number of contiguous stations near the end of the beam. This
yields a set of simultaneous equations which can be solved for the
initially unknown coefficients $D\phi_0$, $D^2\phi_0$, $D^3\phi_0$, ... or
$D\phi_1$, $D^2\phi_1$, $D^3\phi_1$ .... in terms of the quantities ϕ_1, ϕ_2, ϕ_3, ...
Once these coefficients are obtained in this way, the various deri-
vatives of ϕ at any specified station may be evaluated analytically

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from the basic series expression. The result is the finite difference formula of interest.

By introducing an additional term into the original series and retracing the original solution in the usual way, the leading term of the truncation error can readily be found.

The above process has been carried out for all of the conventional finite difference formulas of Table I which involve excessive truncation errors. The revisions are always such as to render all truncation errors consistently of order h^2. The results of this revision are summarized in Table II. Notice that every formula in Table I whose error term is initially of order h^2 reappears in Table II without change. Only the faulty formulas have been revised.

An interesting detail appears in Eq. (8) of Table II. The truncation error term is seen to involve h^2 but the numerical coefficient happens to vanish. Hence the true truncation error degenerates to order h^3 in this particular instance, which is exceptional. It might be supposed that this circumstance would permit a reduction of band width here but such is not the case for if the band width be reduced by one, the resulting truncation error is found to be of order h not of order h^2. Hence Eq. (8) must be retained as written.

Table II Finite Difference Formulas of Consistent Second Order
Truncation Error

Central Differences

$$D\phi_i \doteq \frac{1}{h}\left[0 - \frac{1}{2}\phi_{i-1} + 0 + \frac{1}{2}\phi_{i+1} + 0\right] - \frac{1}{6}h^2 D^3\phi_i + \ldots \tag{1}$$

$$D^2\phi_i \doteq \frac{1}{h^2}\left[0 + \phi_{i-1} - 2\phi_i + \phi_{i+1} + 0\right] - \frac{1}{12}h^2 D^4\phi_i + \ldots \tag{2}$$

$$D^3\phi_i \doteq \frac{1}{h^3}\left[-\frac{1}{2}\phi_{i-2} + \phi_{i-1} + 0 - \phi_{i+1} + \frac{1}{2}\phi_{i+2}\right] - \frac{1}{4}h^2 D^5\phi_i + \ldots \tag{3}$$

$$D^4\phi_i \doteq \frac{1}{h^4}\left[\phi_{i-2} - 4\phi_{i-1} + 6\phi_i - 4\phi_{i+1} + \phi_{i+2}\right] - \frac{1}{6}h^2 D^6\phi_i + \ldots \tag{4}$$

Clamped End

$$D\phi_1 \doteq \frac{1}{h}\left[0 + \frac{1}{2}\phi_2 + 0\right] \qquad\qquad - \frac{1}{6}h^2 D^3\phi_1 + \ldots \tag{5}$$

$$D^2\phi_1 \doteq \frac{1}{h^2}\left[-2\phi_1 + \phi_2 + 0\right] \qquad\qquad - \frac{1}{12}h^2 D^4\phi_1 + \ldots \tag{6}$$

$$D^3\phi_1 \doteq \frac{1}{h^3}\left[-3\phi_1 + 0 + \frac{1}{3}\phi_3\right] \qquad\qquad - \frac{3}{20}h^2 D^5\phi_{\phi_1} + \ldots \tag{7}*$$

$$D^4\phi_1 \doteq \frac{1}{h^4}\left[16\phi_1 - 9\phi_2 + \frac{8}{3}\phi_3 - \frac{1}{4}\phi_4\right] + (0)h^2 D^6\phi_{\phi_1} + \ldots \tag{8}*$$

Table II   Cont'd

Simply Supported End

$$D\phi_1 \doteq \frac{1}{h} \left[0 + \frac{1}{2}\phi_2 + 0 \right] \qquad\qquad - \frac{1}{6} h^2 D^3\phi_1 + \ldots \qquad (9)$$

$$D^2\phi_1 \doteq \frac{1}{h^2} \left[-2\phi_1 + \phi_2 + 0 \right] \qquad\qquad - \frac{1}{12} h^2 D^4\phi_1 + \ldots \qquad (10)$$

$$D^3\phi_1 \doteq \frac{1}{h^3} \left[\frac{3}{11}\phi_1 - \frac{9}{11}\phi_2 + \frac{5}{11}\phi_3 \right] \qquad + \frac{9}{44} h^2 D^5\phi_0 + \ldots \qquad (11)*$$

$$D^4\phi_1 \doteq \frac{1}{h^4} \left[\frac{32}{5}\phi_1 - \frac{27}{5}\phi_2 + \frac{8}{5}\phi_3 - \frac{1}{10}\phi_4 \right] \qquad - \frac{2}{25} h^2 D^6\phi_0 + \ldots \qquad (12)*$$

Free End
    At i = 1

$$D\phi_1 \doteq \frac{1}{h} \left[-\phi_1 + \phi_2 + 0 \right] \qquad\qquad - \frac{1}{6} h^2 D^3\phi_1 + \ldots \qquad (13)$$

$$D^2\phi_1 = 0 \qquad\qquad (14)$$

$$D^3\phi_1 = 0 \qquad\qquad (15)$$

$$D^4\phi_1 \doteq \frac{1}{h^4} \left[4.2353\phi_1 - 9.1765\phi_2 + 5.6471\phi_3 - 0.7059\phi_4 \right]$$

$$+ 0.2255 h^2 D^6\phi_1 + \ldots \qquad (16)*$$

Table II Cont'd

<u>At i = 2</u>

$$D\phi_2 \doteq \frac{1}{h}\left[\frac{1}{2}\phi_1 + 0 + \frac{1}{2}\phi_3 + 0\right] \qquad\qquad -\frac{1}{6}h^2D^3\phi_2 + \ldots \qquad (17)$$

$$D^2\phi_2 \doteq \frac{1}{h^2}\left[\phi_1 - 2\phi_2 + \phi_3 + 0\right] \qquad\qquad -\frac{1}{12}h^2D^4\phi_2 + \ldots \qquad (18)$$

$$D^3\phi_2 \doteq \frac{1}{h^3}\left[-15\phi_1 + \frac{3}{2}\phi_2 - \frac{1}{9}\phi_3 + 0\right] \qquad +\frac{7}{60}h^2D^5\phi_1 + \ldots \qquad (19)*$$

$$D^4\phi_2 \doteq \frac{1}{h^4}\left[-1.6471\phi_1 + 4.2353\phi_2 - 3.5294\phi_3 + 0.9412\phi_4\right]$$

$$+ 1.1451\,h^2D^6\phi_1 + \ldots \qquad (20)*$$

*The asterisks denote those formulas of Table II which differ from the corresponding formulas of Table I.

4. <u>Finite Difference Formulas of Minimum Truncation Error for
Specified Band Width</u>

An interesting and significant feature of Table II is that
the finite difference expressions for the various lower derivatives
usually involve lesser band widths than that required for calcula-
ting $D^4\phi$. The latter is fixed by the requirement that the
truncation error in the approximation for $D^4\phi$ should be of order
h^2 .

A natural question that now arises is whether there be any
possible further advantage in making use of this full available
band width to reduce the truncation errors associated with these
lower derivatives.

This approach is attractive in that it entails no increase
whatever in the overall band width of the final calculation matrix.
Hence it can be accomplished with a negligible increase in the
calculation burden.

On the other hand, the method has the limitation that, although
the accuracy of the lower derivatives is improved, the error in the
overall equation still remains of order h^2 . Hence what we have
here at best is the somewhat uncertain possibility of a marginal
further improvement in accuracy at essentially negligible cost.

In order to permit investigation of this possibility, a further
revision of the finite difference formulas was therefore carried
out based on two principles. The first was to choose a band width
such as to approximate $D^4\phi$ to order h^2 . The second was to
utilize the resulting full band with for estimating all lower

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derivatives. The finite difference formulas obtained in this way, along with their associated truncation errors, are summarized in Table III. Notice that all expressions for $D^4\phi$ are the same in Table III as in Table II. Only the lower derivatives are different.

A further significant feature of Tables II and III is that the expressions for $D^4\phi_1$ at station $i=1$ involve terms in ϕ_1, ϕ_2, ϕ_3, ϕ_4. The first three of these fall within the overall penta-diagonal matrix format which characterizes all interior stations $i = 2, 3, 4, \ldots$ etc. But the term in ϕ_4 falls outside this band at $i = 1$.

Similar considerations also apply at the opposite end, station $i = N$. Thus $D^4\phi_N$ invloves terms in ϕ_{N-3}, ϕ_{N-2}, ϕ_{N-1} and ϕ_N and the term in ϕ_{N-3} falls outside the basic penta-diagonal format.

Fortunately, there is a very simple method to eliminate the term in ϕ_4 from the equation at $i = 1$ and the term in ϕ_{N-3} from the equation at $i = N$. This restores the strictly penta-diagonal format which is so convenient and efficient in the final numerical solution. The details of the method are explained in a later section of this paper.

Table III   Finite Difference Formulas of Minimum Truncation Error
for Specified Band Width**

Central Differences

$$D\phi_i \doteq \frac{1}{h}\left[\frac{1}{12}\phi_{i-2} - \frac{2}{3}\phi_{i-1} + 0 + \frac{2}{3}\phi_{i+1} - \frac{1}{12}\phi_{i+2}\right] + \frac{1}{30}h^4 D^5\phi_i + \dots \tag{1}*$$

$$D^2\phi_i \doteq \frac{1}{h^2}\left[-\frac{1}{12}\phi_{i-2} + \frac{4}{3}\phi_{i-1} - \frac{5}{2}\phi_i + \frac{4}{3}\phi_{i+1} - \frac{1}{12}\phi_{i+2}\right]$$
$$+ \frac{1}{90}h^4 D^6\phi_i + \dots \tag{2}*$$

$$D^3\phi_i \doteq \frac{1}{h^3}\left[-\frac{1}{2}\phi_{i-2} + \phi_{i-1} + 0 - \phi_{i+1} + \frac{1}{2}\phi_{i+2}\right] - \frac{1}{4}h^2 D^5\phi_i + \dots \tag{3}$$

$$D^4\phi_i \doteq \frac{1}{h^4}\left[+\phi_{i-2} - 4\phi_{i-1} + 6\phi_i - 4\phi_{i+1} + \phi_{i+2}\right] - \frac{1}{6}h^2 D^6\phi_i + \dots \tag{4}$$

Clamped End

$$D\phi_1 \doteq \frac{1}{h}\left[\frac{1}{6}\phi_1 + \frac{3}{4}\phi_2 - \frac{1}{6}\phi_3 + \frac{1}{48}\phi_4\right] \qquad -\frac{1}{120}h^5 D^6\phi_o + \dots \tag{5}*$$

$$D^2\phi_1 \doteq \frac{1}{h^2}\left[-\frac{10}{3}\phi_1 + \frac{7}{4}\phi_2 - \frac{2}{9}\phi_3 + \frac{1}{48}\phi_4\right] \qquad -\frac{1}{360}h^4 D^6\phi_o + \dots \tag{6}*$$

$$D^3\phi_1 \doteq \frac{1}{h^3}\left[0 - \frac{9}{4}\phi_2 + \frac{4}{3}\phi_3 - \frac{3}{16}\phi_4\right] \qquad +\frac{1}{12}h^3 D^6\phi_o + \dots \tag{7}*$$

$$D^4\phi_1 \doteq \frac{1}{h^4}\left[16\phi_1 - 9\phi_2 + \frac{8}{3}\phi_3 - \frac{1}{4}\phi_4\right] \qquad + (0)h^2 D^6\phi_o + \dots \tag{8}$$

Table III   Cont'd

Simply Supported End

$$D\phi_1 \doteq \frac{1}{h}\left[-\frac{47}{150}\phi_1 + \frac{93}{100}\phi_2 - \frac{11}{50}\phi_3 + \frac{17}{600}\phi_4\right] - \frac{111}{9000}h^2D^6\phi_0 + \dots \quad (9)*$$

$$D^2\phi_1 \doteq \frac{1}{h^2}\left[-\frac{38}{15}\phi_1 + \frac{29}{20}\phi_2 - \frac{2}{15}\phi_3 + \frac{1}{120}\phi_4\right] + \frac{7}{1800}h^4D^6\phi_0 + \dots \quad (10)*$$

$$D^3\phi_1 \doteq \frac{1}{h^3}\left[\frac{12}{5}\phi_1 - \frac{63}{20}\phi_2 + \frac{8}{5}\phi_3 - \frac{9}{40}\phi_4\right] + \frac{558}{5400}h^3D^6\phi_0 + \dots \quad (11)*$$

$$D^4\phi_1 \doteq \frac{1}{h^4}\left[\frac{32}{5}\phi_1 - \frac{27}{5}\phi_2 + \frac{8}{5}\phi_3 - \frac{1}{10}\phi_4\right] - \frac{2}{25}h^2D^6\phi_0 + \dots \quad (12)$$

Free End

At station $i = 1$

$$D\phi_1 \doteq \frac{1}{h}\left[-1.1274\phi_1 + 1.2706\phi_2 - 0.1588\phi_3 + 0.0157\phi_4\right]$$
$$- 0.0035\ h^5D^6\phi_1 + \dots (13)*$$

$$D_2^2\phi_1 \doteq 0 \quad (14)$$

$$D^3\phi_1 \doteq 0 \quad (15)$$

$$D^4\phi_1 \doteq \frac{1}{h^4}\left[4.2353\phi_1 - 9.1765\phi_2 + 5.6471\phi_3 - 0.7059\phi_4\right]$$
$$+ 2255\ h^2D^6\phi_1 + \dots (16)$$

At station $i = 2$

$$D\phi_2 \doteq \frac{1}{h}\left[-\frac{2}{3}\phi_1 + \frac{3}{10} + \frac{4}{10}\phi_3 - \frac{1}{30}\phi_4\right] + 0.0106\ h^5D^6\phi_1 \quad (17)*$$

$$D^2\phi_2 \doteq \frac{1}{h^2}\left[1.1373\phi_1 - 2.3529\phi_2 + 1.2941\phi_3 - 0.0784\phi_4\right]$$
$$+ 0.0740\ h^4D^6\phi_1 + \dots (18)*$$

Table III Cont'd

$$D^3\phi_2 \doteq \frac{1}{h^3} \left[1.2941\phi_1 - 2.4706\phi_2 + 1.0588\phi_3 + 0.1177\phi_4 \right]$$
$$+ 0.3765 \; h^3 D^6\phi_1 + \ldots \quad (19)$$

$$D^4\phi_2 \doteq \frac{1}{h^4} \left[- 1.6471\phi_1 + 4.2353\phi_2 - 3.5294\phi_3 + 0.9412\phi_4 \right]$$
$$+ 1.1451 \; h^2 D^6\phi_1 + \ldots \quad (20)$$

*The asterisks denote those formulas of Table III which differ
from the corresponding formulas of Table II.

**In this table the band width is always chosen so as to give
a truncation error of order h^2 for $D^4\phi$. The truncation errors
of the other derivatives are then fixed accordingly.

5. Difference Formulas at Right Boundary

It has been convenient to present the boundary formulas in Tables I, II and III specifically for the left end of the span. The corresponding formulas for the right end can readily be inferred from the foregoing results from considerations of symmetry. First we replace ϕ_1, ϕ_2, ϕ_3, and ϕ_4 by the corresponding quantities ϕ_N, ϕ_{N-1}, ϕ_{N-2} and ϕ_{N-4} respectively. In this connection it is also convenient to reverse the order of corresponding terms from left to right so that the terms still appear in order of increasing index. Secondly, we must reverse all signs for the derivatives of odd order.

We illustrate these rules by applying them to a clamped end, as given originally in Eqs. (5) through (8) of Table I. Thus for the right end clamped

$$D\phi_N \doteq \frac{1}{h} \left[0 - \frac{1}{2} \phi_{N-1} + 0 \right] + \frac{1}{6} h^2 D^3 \phi_N \tag{5.1}$$

$$D^2\phi_N \doteq \frac{1}{h^2} \left[0 + \phi_{N-1} - 2\phi_N \right] - \frac{1}{12} h^2 D^4 \phi_N \tag{5.2}$$

$$D^3\phi_N \doteq \frac{1}{h^3} \left[-\frac{1}{2} \phi_{N-2} + \phi_{N-1} - \frac{1}{2} \phi_N \right] + \frac{1}{6} D^3 \phi_{N+1} \tag{5.3}$$

$$D^4\phi_N \doteq \frac{1}{h^4} \left[\phi_{N-2} - 4\phi_{N-1} + 7\phi_N \right] - \frac{1}{3} \frac{D^3\phi_{N+1}}{h} \tag{5.4}$$

The same method can of course be applied to any of the other results developed in this paper to adapt them to the right end of the span.

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6. <u>Reduction to Penta-Diagonal Format at Boundaries</u>

If the basic relation, Eq. (1.1), be expressed in terms of
the conventional finite difference formulas of Table I, the re-
sulting calculation matrix is strictly penta-diagonal. It has
earlier been pointed out, however, that if Eq. (1.1) be expressed
in terms of the revised finite difference formulas either of Table
II or of Table III, the resulting calculation matrix, while still
penta-diagonal over all interior rows, falls outside the penta-
diagonal format by one place in the first row $i = 1$ and last row
$i = N$. The purpose of this section is to show how these two
minor deviations can easily be rectified.

Let us consider the left end of the span first. More speci-
fically, consider the final matrix equations at stations $i = 1$
and $i = 2$. These will reduce to the form

$$\alpha_{11} \phi_1 + \alpha_{12} \phi_2 + \alpha_{13} \phi_3 + \alpha_{14} \phi_4 = g_1 \qquad (6.1)$$

$$\alpha_{21} \phi_1 + \alpha_{22} \phi_2 + \alpha_{23} \phi_3 + \alpha_{24} \phi_4 = g_2 \qquad (6.2)$$

where the coefficients α_{ij} and the quantities g_1 and g_2
are all known constants in any particular case.

We can eliminate ϕ_4 between these two equations by multi-
plying Eq. (6.1) through by α_{24} and Eq. (6.2) by $- \alpha_{14}$, then
adding. These operations give rise to the following auxiliary
constants which can be readily computed as definite numbers, namely,

$$\alpha'_{11} = \alpha_{24}\alpha_{11} - \alpha_{14}\alpha_{21}$$

$$\alpha'_{12} = \alpha_{24}\alpha_{12} - \alpha_{14}\alpha_{22}$$

$$\alpha'_{13} = \alpha_{24}\alpha_{13} - \alpha_{14}\alpha_{23}$$

$$g'_1 = \alpha_{24}g_1 - \alpha_{14}g_2 \tag{6.3}$$

In this way we finally obtain the relation

$$\alpha'_{11}\phi_1 + \alpha'_{12}\phi_2 + \alpha'_{13}\phi_3 = g'_1 \tag{6.4}$$

Eq. (6.1) of the original matrix can now be replaced by Eq. (6.4). This substitution reduces the band width at station $i = 1$ such as to fit the matrix into the required penta-diagonal format.

A corresponding correction can also be made at station $i = N$.

Notice that these operations, which reduce the band width in the first and last lines of the final calculation matrix as required, are very simple and do not adversely affect the truncation error of the final system of equations.

7. Example: Uniformly Loaded Beam with Clamped Ends

Two examples have been worked out to illustrate the improvement in accuracy that occurs when the revised difference formulas of Table II or Table III are used in place of the conventional formulas of Table I. In this section we consider a beam of uniform stiffness with clamped ends under a uniform distributed load. In the next section we consider a beam of variable stiffness with simply supported ends under a uniform distributed load.

For the uniform beam, the nondimensional governing equation reduces to

$$D^4\phi = 1 \tag{7.1}$$

and the boundary conditions are for clamped ends

$$\left.\begin{array}{l} \phi = 0 \\ D\phi = 0 \end{array}\right\} \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 1 \tag{7.2}$$

The exact analytical solution of these equations is simply

$$\phi_e = \frac{1}{24} x^2 (1-x)^2 \tag{7.3}$$

In the finite difference solution, only half the beam need be considered because of symmetry. Let index M denote the station at mid-span. The central difference expressions at stations i = M-1 and i = M may be simplified by the substitutions

$$\phi_{M+1} = \phi_{M-1}$$
$$\phi_{M+2} = \phi_{M-2} \tag{7.4}$$

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The conventional finite difference matrix for this case can now be written almost by inspection. For the first row $i = 1$ we simply utilize Eq. (8) of Table I. For the interior rows $i = 2, 3, 4, \ldots$ we use Eq. (4) of Table I. The matrix coefficients for the last two rows $i = M-1$ and $i = M$ are easily determined by making use of Eqs. (7.4) in connection with Eq. (4) of the table.

The following matrix is thereby obtained.

$$\frac{1}{h^4}
\begin{bmatrix}
7 & -4 & 1 & & & & & & \\
-4 & 6 & -4 & 1 & & & & 0 & \\
1 & -4 & 6 & -4 & 1 & & & & \\
 & 1 & -4 & 6 & -4 & 1 & & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & & & 1 & -4 & 6 & -4 & 1 & \\
 & 0 & & & 1 & -4 & 6 & -4 & 1 \\
 & & & & & 1 & -4 & 7 & -4 \\
 & & & & & & 2 & -8 & 6
\end{bmatrix}
\begin{Bmatrix}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4 \\
\cdot \\
\phi_{M-3} \\
\phi_{M-2} \\
\phi_{M-1} \\
\phi_M
\end{Bmatrix}
=
\begin{Bmatrix}
1 \\
1 \\
1 \\
1 \\
\cdot \\
1 \\
1 \\
1 \\
1
\end{Bmatrix}
\qquad (7.5)$$

The matrix for the revised difference scheme of either Table II or Table III is identical to Eq. (7.5) except for the first row. Using the revised difference formula shown in Eq. (8) of Table II or Table III gives the following improved approximation for the first row, namely,

$$\frac{1}{h^4} \left[16\phi_1 - 9\phi_2 + \frac{8}{3} \phi_3 - \frac{1}{4} \phi_4 \right] = 1 \qquad (7.6)$$

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The second row remains unchanged, namely,

$$\frac{1}{h^4} \left[- 4\phi_1 + 6\phi_2 - 4\phi_3 + \phi_4 \right] = 1 \qquad (7.7)$$

Multiplying Eq.(7.7) through by $+\frac{1}{4}$ and adding it to Eq. (7.6) eliminates ϕ_4 and gives

$$\frac{1}{h^4} \left[15\phi_1 - \frac{15}{2} \phi_2 + \frac{5}{3} \phi_3 \right] = \frac{5}{4} \qquad (7.8)$$

Eq. (7.8) is the contracted version of the revised first row that is inserted into the final matrix. This version is just as accurate as Eq. (7.6) but has the advantage that it fits strictly within the penta-diagonal format.

Solutions have been obtained in double precision to Eq. (7.5) for both the conventional first row and for the revised first row given by Eq. (7.8). Results were calculated for three values of the important dimensionless mesh size parameter h, namely, $h = 1/10$, $h = 1/20$ and $h = 1/40$. The calculations were performed in double precision on the IBM 360/67 computer at the Naval Postgraduate School.

As convenient measures of the overall relative error of the final numerical solution, the following two parameters are used, namely,

$$\epsilon = \frac{\left| \phi - \phi_e \right|_{max}}{\left| \phi_e \right|_{x=1/2}} \qquad (7.9)$$

and

$$\delta = \frac{\left| D^2\phi - D^2\phi_e \right|_{max}}{\left| D^2\phi_e \right|_{x=0}} \qquad (7.10)$$

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In beam problems ε expresses the maximum relative error in the calculated deflection while δ expresses the maximum relative error in the calculated curvature, and hence in the associated bending stresses.

Curves of ε and δ versus h are shown in Fig. 1 corresponding to the conventional difference formulas of Table I and to the revised formulas of Tables II and III.

The results are quite interesting. They confirm that both error indices ε and δ are greatest for the conventional difference formulas of Table I. At the other extreme, both ε and δ vanish identically for the difference formulas of Table III. Of course, such a complete elimination of all truncation error must be regarded as exceptional. It can only happen for those cases, like the present example, in which the true deflection curve happens to be a quartic. The curve is a quartic for any uniform beam under uniform load, irrespective of the boundary conditions.

The revised finite difference formulas of Table II are seen to produce results in this case intermediate between those given by Tables I and III. Thus ε vanishes but δ does not. By an interesting coincidence, the formulas of Tables I and II happen to give identical curves of δ versus h in this example. Inspection of the detailed solutions reveal, however, that the actual truncation errors represented by Table II are precisely equal in magnitude but opposite in sign to those produced by the conventional formulas of Table I! No specific reason is known for this curious and striking co-incidence.
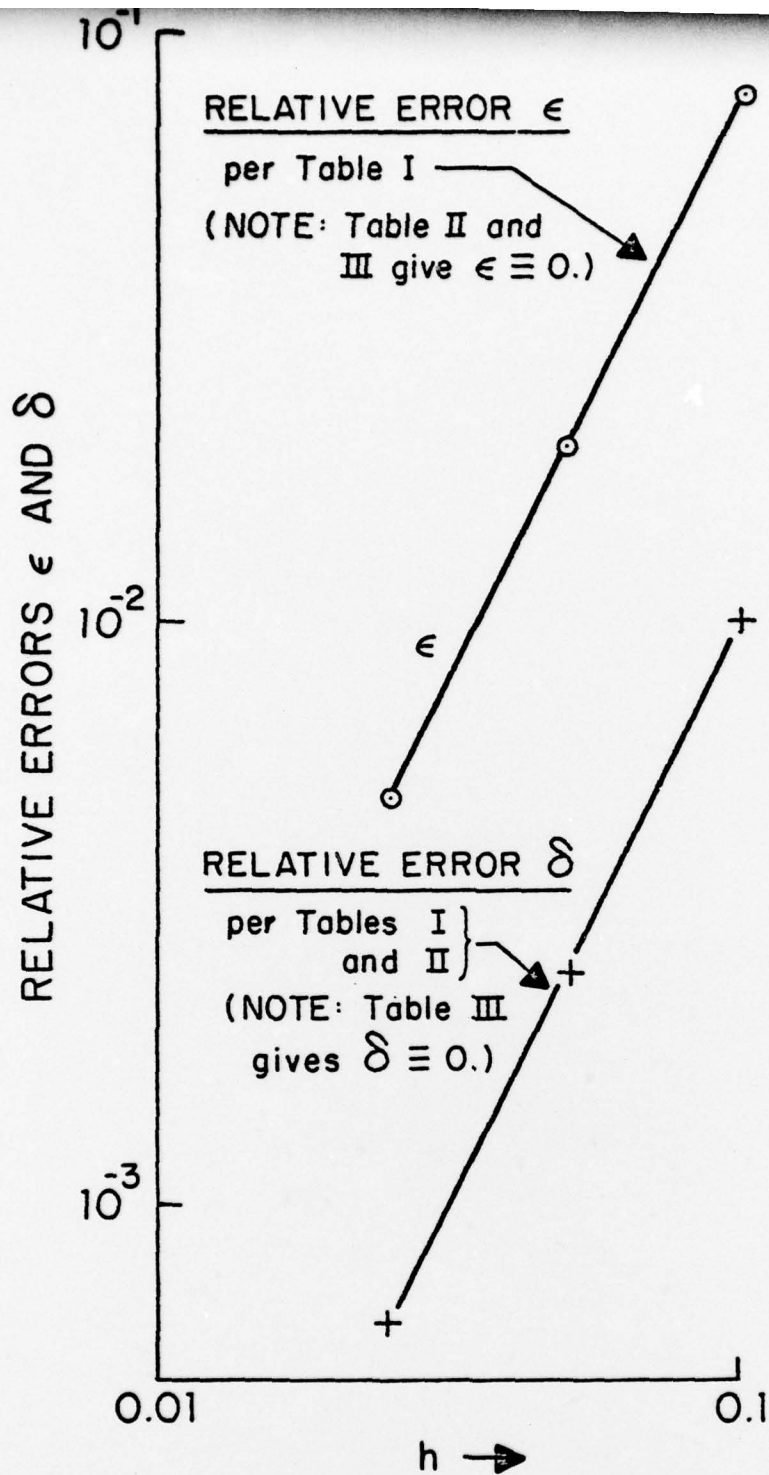
FIG. 1: ERRORS OF NUMERICAL SOLUTION FOR BEAM OF UNIFORM STIFFNESS WITH CLAMPED ENDS.

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8. Example: Beam of Variable Stiffness with Simply Supported Ends

In this section we consider a uniformly loaded beam of variable stiffness, simply supported at both ends. Specifically, we choose a beam governed by the nondimensional equation

$$D^2 (x^3 D^2 \phi) = x^3 D^4 \phi + 6x^2 D^3 \phi + 6x D^2 \phi = 1 \tag{8.1}$$

over the range

$$1 \leq x \leq 2 \tag{8.2}$$

The boundary conditions for simply supported ends are

$$\left. \begin{array}{l} \phi = 0 \\ D^2 \phi = 0 \end{array} \right\} \quad \text{at} \quad x = 1 \quad \text{and} \quad x = 2 \tag{8.3}$$

The exact solution for ϕ is

$$\phi_e = \frac{1}{4} (10 \ln 2 - 3)(1-x) + \frac{1}{2} \left[\frac{1}{x} + (3+x) \ln x - x \right] \tag{8.4}$$

Next we must obtain the conventional finite difference matrix corresponding to Table I. It is convenient in this case to multiply Eq. (8.1) through by h^4 and to introduce the notation

$$x_i = 1 + i h \quad i = 1, 2, 3, \ldots \tag{8.5}$$

For all stations other than station $i = 1$ and station $i = N$, the regular central difference formulas of Eqs. (2), (3) and (4) of Table I apply. Upon substituting these equations into Eq. (8.1) and regrouping terms, we can reduce the result to the following format, namely,

$$(x_i^3 - 3x_i^2 h) \, \phi_{i-2} + (- 4x_i^3 + 6x_i^2 h + 6x_i h^2) \, \phi_{i-1}$$

$$+ (6x_i^3 - 12x_i h^2) \, \phi_i + (- 4x_i^3 - 6x_i^2 h + 6x_i h^2) \, \phi_{i+1} \qquad (8.6)$$

$$+ (x_i^3 + 3x_i^2 h) \, \phi_{i+2} = h^4$$

For station $i = 1$, Eqs. (10), (11) and (12) of Table I apply and Eq. (8.1) reduces in this case to

$$(5x_1^3 + 3x_1^2 h - 12x_1 h^2) \, \phi_1 + (- 4x_1^3 - 6x_1^2 h + 6x_1 h^2) \, \phi_2 \qquad (8.7)$$

$$+ (x_1^3 + 3x_1^2 h) \, \phi_3 = h^4$$

For station $i = N$, expressions must be obtained for $D^2 \phi_N$, $D^3 \phi_N$, $D^4 \phi_N$ by analogy with Eqs. (10), (11) and (12) of Table I, using the rules of symmetry summarized in section 5. Then Eq. (8.1) reduces to

$$(x_N^3 - 3x_N^2 h) \, \phi_{N-2} + (- 4x_N^3 + 6x_N^2 h + 6x_N h^2) \phi_{N-1} \qquad (8.8)$$

$$+ (5x_N^3 - 3x_N^2 h - 12x_N h^2) \, \phi_N = h^4$$

The finite difference matrix defined by Eqs. (8.6), (8.7) and (8.8) was solved numerically for various values of the dimensionless mesh size parameter h . Each result was compared with the corresponding exact analytical solution of Eq. (8.4) for the same value of h . The relative error ε , as defined by Eq. (7.9), was calculated for each solution. The second error function δ , as defined in Eq. (7.10), was not calculated for the present example.

The above values of ε were plotted against h . The result is represented in Fig. 2 by the line marked I.

Consider next the revised solution based upon the revised finite difference formulas proposed in Table II. Careful examination of Eq. (8.1) and of Tables I and II discloses that this case differs from the previous one only in the first and last rows, $i = 1$ and $i = N$.

The revised equations for the first and last rows turn out to be as follows.

$$(\frac{32}{5} x_1^3 + \frac{18}{11} x_1^2 h - 12 x_1 h^2) \phi_1$$

$$+ (- \frac{27}{5} x_1^3 - \frac{54}{11} x_1^2 h + 6 x_1 h^2) \phi_2 \qquad (8.9)$$

$$+ (\frac{8}{5} x_1^3 + \frac{30}{11} x_1^2 h) \phi_3 + (- \frac{1}{10} x_1^3) \phi_4 = h^4$$

and

$$(- \frac{1}{10} x_N^3) \phi_{N-3} + (\frac{8}{5} x_N^3 - \frac{30}{11} x_N^2 h) \phi_{N-2}$$

$$\qquad (8.10)$$

$$+ (- \frac{27}{5} x_N^3 + \frac{54}{11} x_N^2 h + 6 x_N h^2) \phi_{N-1} + (\frac{32}{5} x_N^3 - \frac{18}{11} x_N^2 h - 12 x_N h^2) \phi_N = h^4$$

The resulting matrix was then completed, solved and analyzed in the same general fashion as described above for the previous case. The corresponding curve of error ϵ versus mesh size h is shown in Fig. 2 by the line marked II.

The same general pattern was also repeated for the finite difference formulas of Table III. In this case the entire matrix must be modified. The new matrix is defined by the following three governing equations.
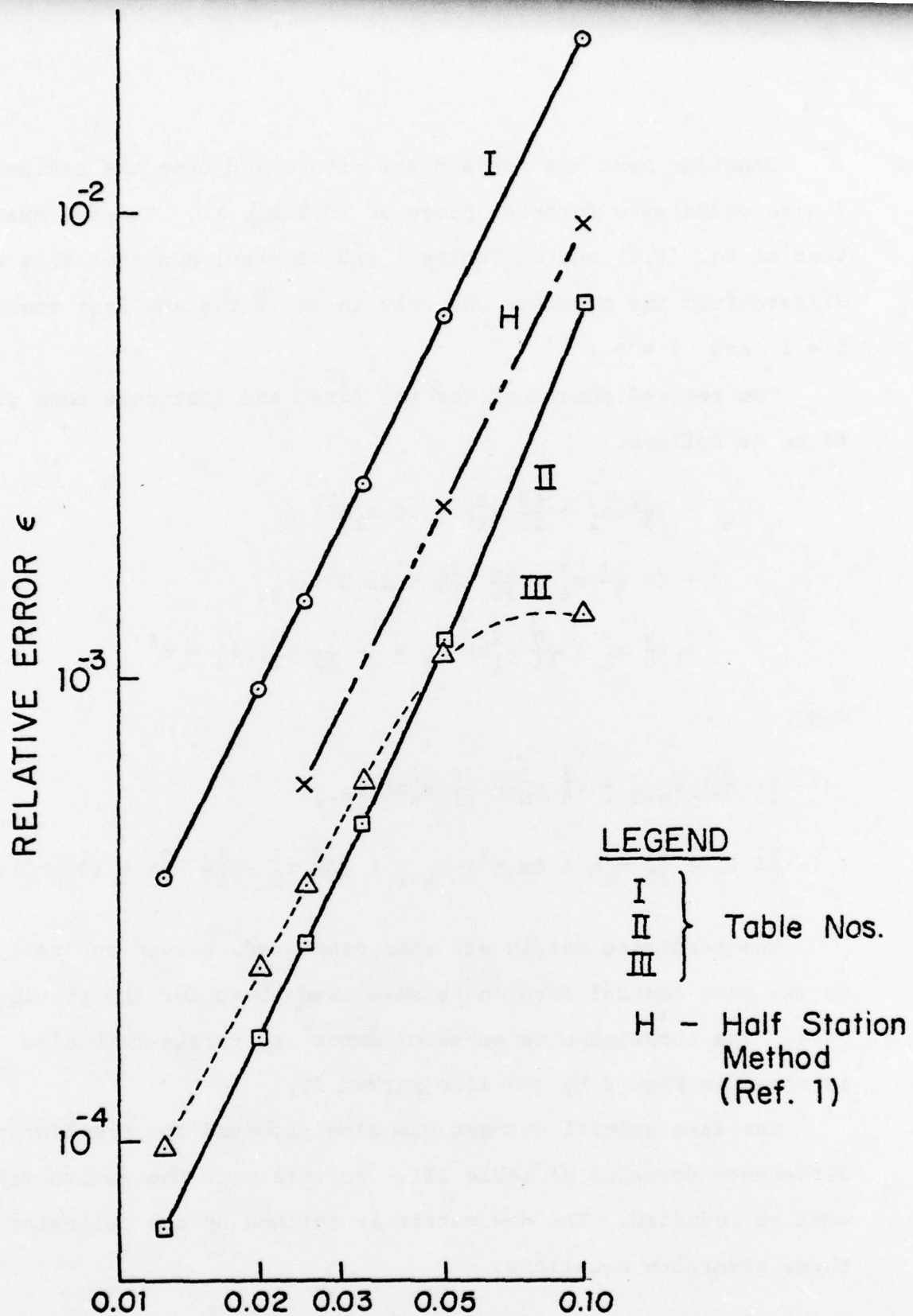
FIG. 2: ERRORS OF VARIOUS APPROXIMATIONS
FOR BEAM OF VARIABLE STIFFNESS
WITH SIMPLY SUPPORTED ENDS.

For all interior rows $2 \leq i \leq (N-1)$ the basic equation is found by substituting Eqs. (2), (3) and (4) of Table III into Eq. (8.1) This reduces to the result

$$(x_i^3 - 3x_i^2 h - \frac{1}{2} x_i h^2) \, \phi_{i-2} + (- 4x_i^3 + 6x_i^2 h + 8x_i h^2) \, \phi_{i-1}$$

$$+ (6x_i^3 - 15 x_i h^2) \, \phi_i + (- 4x_i^3 - 6x_i^2 h + 8x_i h^2) \, \phi_{i+1}$$

$$+ (x_i^3 + 3x_i^2 h - \frac{1}{2} x_i h^2) \, \phi_{i+2} = h^4 \tag{8.11}$$

The result for row $i = 1$ is found by substituting Eqs. (10), (11) and (12) of Table III into Eq. (8.1). This gives

$$(\frac{32}{5} x_1^3 + \frac{72}{5} x_1^2 h - \frac{76}{5} x_1 h^2) \, \phi_1 + (- \frac{27}{5} x_1^3 - \frac{189}{10} x_1^2 h + \frac{87}{10} x_1 h^2) \, \phi_2$$

$$+ (\frac{8}{5} x_1^3 + \frac{48}{5} x_1^2 h - \frac{4}{5} x_1 h^2) \, \phi_3 + (- \frac{1}{10} x_1^3 - \frac{27}{20} x_1^2 h + \frac{1}{20} x_1 h^2) \, \phi_4$$

$$= h^4 \tag{8.12}$$

The corresponding result for station $i = N$ is found from the above by applying the rules of symmetry as explained in section 5. This gives

$$(- \frac{1}{10} x_N^3 + \frac{27}{20} x_N^2 h + \frac{1}{20} x_N h^2) \, \phi_{N-3} + (\frac{8}{5} x_N^3 - \frac{48}{5} x_N^2 h - \frac{4}{5} x_N h^2) \, \phi_{N-2}$$

$$+ (- \frac{27}{5} x_N^3 + \frac{189}{10} x_N^2 h + \frac{87}{10} x_N h^2) \, \phi_{N-1} + (\frac{32}{5} x_N^3 - \frac{72}{5} x_N^2 h - \frac{76}{5} x_N h^2) \phi_N$$

$$= h^4 \tag{8.13}$$

Of course, the above results for rows $i = 1$ and $i = N$ now lie outside the band limits for a penta-diagonal matrix, but they can readily be contracted by the method explained in section 6.

Eqs. (8.11), (8.12) and (8.13) were solved numerically in
the usual way. The resulting curve of ε versus h is shown
in Fig. 2 as the line marked III. The marked change in the error
curve between $h = 0.1$ and 0.05 is due to the fact that the
sign of the error changes at the centerline.

An interesting comparison of the results from the methods
proposed herein can be made with the results of a study using the
"half-station" method presented in Ref. 1. In the half-station
method the finite difference approximations are made before ex-
panding the derivatives in $D^2(x^3 D^2 \phi)$ as opposed to the conven-
tional, or whole-station method, wherein the finite difference
approximations are made after $D^2(x^3 D^2 \phi)$ is totally expanded.
The conventional finite difference approximation for the boundary
condition $D^2 \phi$ was used at the two ends of the beam in Ref. 1.
The relative error ε , as defined by Eq. (7.9), produced by the
half-station method for this example was shown in Ref. 1 to be of
the form

$$\varepsilon \doteq (0.4 \times 10^{-2}/0.4196 \times 10^{-2}) \, h^2 \qquad (8.14)$$

where the value 0.4×10^{-2} is obtained from Fig. 3 of Ref. 1
and 0.4196×10^{-2} is the displacement of the beam at the center-
line. This expression for ε was verified by a numerical
solution of the half-station equations for several values of h .
The error given by Eq. (8.14) is plotted in Fig. 2 as the line
marked H .

Fig. 2 permits some very interesting and useful conclusions to be drawn. Like the previous example, it again confirms that the conventional finite difference formulas of Table I give by far the highest overall errors. The revised formulas of Table II are seen to reduce the above errors by a factor of approximately five. It is interesting that curves II and III cross at approximately $h \approx 0.05$, and that for mesh sizes finer than this, the formulas of Table II give somewhat better results than those of Table III. Inasmuch as the former are also somewhat simpler, there would seldom appear to be much reason to resort to the latter.

Notice too that while the half station method of Ref. 1 is superior to the conventional formulas of Table I, the revised formulas of Table II are in turn superior to the half station method.

We may conclude that the revised finite difference formulas of Table II probably represent the best overall compromise between accuracy and simplicity for most typical problems.

9. References

1. Cyrus, N. J. and Fulton, R. E., "Finite Difference Accuracy in Structural Analysis," _J. Structural Div._ ASCE, Vol. 92, No. ST6, Dec. 1966, pp 459-471.

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